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The quantum under-, critical- and over-damped driven harmonic oscillators

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Abstract

The cases of under-, critical- and over-damping are treated for the quantum driven harmonic oscillator. Following a survey of the classical version, the quantum invariant operator for each case is constructed and their eigenvalues are evaluated. Using these eigenvalues and the Schrödinger equations, the wavefunctions are obtained for each case. From formal path integral theory, their propagators are evaluated and are checked with those obtained by the closed property formed by the complete set of wavefunctions.

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1. Introduction

The damped driven harmonic oscillator provides many opportunities for basic studies and simple models in classical mechanics, perhaps exceeded only by those of the simple harmonic oscillator [1]. This offers a fundamental exercise as well as applications for dissipative systems or systems subject to an external field. There are three general cases for damping motion: under-, critical- and over-damping, which are distinguished through the damping constant and the frequency. Only the case of under-damping results in oscillatory motion. Hence, while much attention has been paid over the past few decades to analysing the quantum damped driven harmonic oscillator [2–5], only for the case of under-damping has the oscillatory motion been treated quantum mechanically.

In this paper, we treat the under-damped driven harmonic oscillator as well as the critical- and over-damped driven harmonic oscillators quantum mechanically. In section 2, we first survey the damped driven harmonic oscillator classically. In a previous paper we showed that there are innumerable classical Hamiltonians or Lagrangians which give the same classical solutions of a given system [6, 7]. In this paper we select the Hamiltonians which are related to what we call the Kanai Hamiltonian, although the equation of motion of a damped driven harmonic oscillator can be found by any other Hamiltonian among innumerable ones [6, 7].

We obtain the eigenfunction of the quantum invariant operator and the Schrödinger solution for the three cases of the damped driven harmonic oscillator. In a previous paper we proved that the quantum Hamiltonian is obtained from the classical Hamiltonian by replacing the canonical variables with their corresponding quantum operators [7], and we use that approach to select the quantum Hamiltonian corresponding to a classical Hamiltonian. We evaluate the quantum invariant operator whose time derivative is zero. Although there are innumerable kinds of this operator, we select a quadratic one. It is well known that the Schrödinger solution of a system can be found by the eigenfunctions of the invariant operator by considering phase factors, which we do for the three cases.

In section 3, we evaluate the propagator for the three cases of the damped driven harmonic oscillator. The propagator of the quadratic Hamiltonian can be obtained as the product of the exponentiated phase, composed of the classical action, and its amplitude [8]. The result of the classical system in section 2 gives the classical action for the three cases. With this result, we evaluate the propagator for the three cases of the damped driven harmonic oscillator. The closed property formed by the wavefunctions also gives the propagator [8]. We check the propagator with that obtained by this method. The summary and conclusions are presented in section 4. Finally, for supporting material, in the appendix we calculate the Green functions for both one- and two-point boundary conditions for the above three cases; with these functions, we review the general and particular solutions of the three cases, and with an example, we examine the solution after a sufficiently long time compared to the inverse damping parameter.

2. Quantum invariant operator and wavefunction of three cases of the damped driven harmonic oscillator

Although there are innumerable classical Hamiltonians of the damped driven harmonic oscillator [7], among them we choose the Hamiltonian for the system as

$$H = \mathrm{e}^{-\alpha t}\frac{p^2}{2m} + \mathrm{e}^{\alpha t}\frac{m}{2}({\omega_0}^2 x^2 - 2f(t)x). \tag{1}$$

The corresponding Lagrangian is

$$L = \mathrm{e}^{\alpha t}\frac{m}{2}(\dot{x}^2 - {\omega_0}^2 x^2 + 2f(t)x). \tag{2}$$

The canonical momentum conjugate to x is

$$p = \mathrm{e}^{\alpha t} m\dot{x} \tag{3}$$

and the classical equation of motion of the system is

$$\ddot{x} + \alpha\dot{x} + {\omega_0}^2 x = f(t). \tag{4}$$

We know that there are three general cases of the solution of (4), that is, the case of the under-damped driven harmonic oscillator

$$-\omega^2 \equiv \frac{\alpha^2}{4} - {\omega_0}^2 < 0 \tag{5}$$

the critical-damped driven harmonic oscillator

$$\frac{\alpha^2}{4} - {\omega_0}^2 = 0 \tag{6}$$

and the over-damped driven harmonic oscillator

$$\beta^2 = \frac{\alpha^2}{4} - {\omega_0}^2 > 0. \tag{7}$$

The general solution of (4) for the three cases is presented in the appendix.

The Schrödinger equation of the system is

$$\mathrm{i}\hbar \frac{\partial \Psi(x,t)}{\partial t} = \hat{H}(\hat{p},\hat{x},t)\Psi(x,t) \tag{8}$$

where $\hat{H}(\hat{p},\hat{x},t)$ is the quantum Hamiltonian, which is obtained by replacing the canonical variables of the classical Hamiltonian, equation (1), with the corresponding quantum operator [7]. The quantum invariant operator satisfies

$$\mathrm{i}\hbar \frac{\partial \hat{I}}{\partial t} + [\hat{I},\hat{H}] = 0. \tag{9}$$

The eigenfunctions of the invariant operator $\hat{I}(\hat{p},\hat{x},t)$, defined as

$$\hat{I}(\hat{p},\hat{x},t)\phi(x,t) = \lambda\phi(x,t) \tag{10}$$

are related to the general solutions of the Schrödinger equation, (8), for the discrete eigenvalue as

$$\Psi(x,t) = \sum_\lambda c_\lambda \Psi_\lambda(x,t) = \sum_\lambda c_\lambda \mathrm{e}^{\gamma_\lambda(t)}\phi_\lambda(x,t) = \sum_n c_n \mathrm{e}^{\gamma_n(t)}\phi_n(x,t) \tag{11}$$

and for the continuous eigenvalue as

$$\Psi(x,t) = \int \mathrm{d}\lambda \Psi(\lambda,x,t) = \int \mathrm{d}\lambda\, \mathrm{e}^{\gamma(\lambda,t)}\phi(x,t,\lambda). \tag{12}$$

Here, c_λ is an arbitrary constant, not dependent on λ; γ_λ and $\gamma(\lambda)$ are a constant and function to be determined; and c_n is related to c_λ, where the subscript n is the number within the sequence of discrete energy eigenvalues [6].

There are an innumerable number of invariant operators which satisfy (10) [7]. Among them, we are interested in the quadratic invariant operator. Using equations (8) and (9), the quadratic invariant operators of the under-, critical- and the over-damped driven harmonic oscillators can be obtained as [6]

$$\hat{I}(\hat{p},\hat{x},t) = \frac{1}{2}\left[\frac{1}{m}\mathrm{e}^{-\alpha t}(\hat{p}-p_\mathrm{p})^2 + m\mathrm{e}^{\alpha t}\omega_0^2(\hat{x}-x_\mathrm{p})^2 + \frac{\alpha}{2}\left[(\hat{x}-x_\mathrm{p})(\hat{p}-p_\mathrm{p}) + (\hat{p}-p_\mathrm{p})(\hat{x}-x_\mathrm{p})\right]\right] \tag{13}$$

$$\hat{I}(\hat{p},\hat{x},t) = \frac{1}{2m}\left[\frac{m\alpha}{2}\mathrm{e}^{\frac{\alpha}{2}t}(\hat{x}-x_\mathrm{p}) + \mathrm{e}^{-\frac{\alpha}{2}t}(\hat{p}-p_\mathrm{p})\right]^2 \tag{14}$$

and

$$\hat{I}(\hat{p},\hat{x},t) = \frac{1}{2}\left[\frac{1}{m}\mathrm{e}^{-\alpha t}(\hat{p}-p_\mathrm{p})^2 + m\omega_0^2\mathrm{e}^{\alpha t}(\hat{x}-x_\mathrm{p})^2 + \frac{\alpha}{2}\left[(\hat{x}-x_\mathrm{p})(p-p_\mathrm{p}) + (\hat{p}-p_\mathrm{p})(\hat{x}-x_\mathrm{p})\right]\right] \tag{15}$$

respectively, where $x_\mathrm{p} = x_\mathrm{p}(t)$ and $p_\mathrm{p} = p_\mathrm{p}(t)$ are the particular solutions of the classical equation of motion, equation (4), and its corresponding canonical momentum, respectively. We treat these in the appendix.

First, let us obtain the Schrödinger solution for the under-damped driven harmonic oscillator. The invariant operator (13) can be represented by the creation and the annihilation operators as [6]

$$\hat{I}(\hat{a},\hat{a}^\dagger) = \hbar\omega(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}) \tag{16}$$

where

$$\hat{a} = \left(\frac{\mathrm{e}^{-\alpha t}}{2m\hbar\omega}\right)^{1/2}\left[m\mathrm{e}^{\alpha t}\left(\omega + \mathrm{i}\frac{\alpha}{2}\right)(\hat{x} - x_\mathrm{p}) + \mathrm{i}(\hat{p} - p_\mathrm{p})\right] \tag{17}$$

and

$$\hat{a}^\dagger = \left(\frac{\mathrm{e}^{-\alpha t}}{2m\hbar\omega}\right)^{1/2}\left[m\mathrm{e}^{\alpha t}\left(\omega - \mathrm{i}\frac{\alpha}{2}\right)(\hat{x} - x_\mathrm{p}) - \mathrm{i}(\hat{p} - p_\mathrm{p})\right]. \tag{18}$$

If $[\hat{x},\hat{p}] = \mathrm{i}\hbar$ holds, $\hat{a}$ in (17) and $\hat{a}^\dagger$ in (18) naturally satisfy

$$[\hat{a},\hat{a}^\dagger] = 1. \tag{19}$$

We know that the ground eigenfunction of (16), denoted by ϕ_0, satisfies [6]

$$\hat{a}\phi_0(x,t) = 0 \tag{20}$$

and the excited eigenfunction can be obtained by

$$\phi_n(x,t) = \frac{1}{\sqrt{n!}}\,\hat{a}^{\dagger n}\phi_0(x,t). \tag{21}$$

Thus, with (17), (18) and (20), (21) gives the eigenfunction of the invariant operator (13) as

$$\phi_n(x,t) = \left(\frac{\sqrt{m\omega/\pi\hbar}}{2^n n!}\right)^{1/2}\mathrm{e}^{\frac{\alpha}{4}t}\,H_n\left[\sqrt{\frac{m\omega}{\hbar}}\,\mathrm{e}^{\frac{\alpha}{2}t}(x - x_\mathrm{p})\right]$$
$$\times\exp\left[-\frac{m\omega}{2\hbar}\mathrm{e}^{\alpha t}(x - x_\mathrm{p})^2\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{m\alpha}{4}\mathrm{e}^{\alpha t}(x - x_\mathrm{p})^2\right]\mathrm{e}^{\mathrm{i}\frac{p_\mathrm{p}}{\hbar}x}. \tag{22}$$

Substitution of (22) with (11) into (8) gives

$$\gamma_n(t) = -\mathrm{i}\omega\left(\frac{1}{2} + n\right)(t - t_0) - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t\;\mathrm{e}^{\alpha t}\left(\frac{m}{2}\dot{x}_\mathrm{p}^2 - \frac{m\omega_0^2}{2}x_\mathrm{p}^2\right) \tag{23}$$

from which we obtain the exact wavefunction of the nth state of the system as

$$\Psi_n(x,t) = \exp\left[-\mathrm{i}\omega\left(\frac{1}{2} + n\right)(t - t_0) - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\mathrm{d}t\;\mathrm{e}^{\alpha t}\left(\frac{m}{2}\dot{x}_\mathrm{p}^2 - \frac{m\omega_0^2}{2}x_\mathrm{p}^2\right)\right]$$
$$\times\left(\frac{\sqrt{m\omega/\pi\hbar}}{2^n n!}\right)^{1/2}\mathrm{e}^{\frac{\alpha}{4}t}\,H_n\left(\sqrt{\frac{m\omega}{\hbar}}\,\mathrm{e}^{\frac{\alpha}{2}t}(x - x_\mathrm{p})\right)$$
$$\times\exp\left[-\frac{m\omega}{2\hbar}\mathrm{e}^{\alpha t}(x - x_\mathrm{p})^2\right]\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{m\alpha}{4}\mathrm{e}^{\alpha t(x - x_\mathrm{p})^2}\right]\mathrm{e}^{\mathrm{i}\frac{p_\mathrm{p}}{\hbar}x} \tag{24}$$

where t_0 within the phase is the starting time of the driving force. The phase of the wavefunction is represented by the classical action composed of the classical particular solution. This means that the phase of the wavefunction depends on the past time.

From (17) and (18), we obtain expressions for $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ as

$$\hat{x} = \left(\frac{\hbar}{2m\omega}\right)^{1/2}\mathrm{e}^{-\frac{\alpha}{2}t}\left(\hat{a}^\dagger + \hat{a}\right) + x_\mathrm{p} \tag{25}$$

$$\hat{p} = -\mathrm{i}\left(\frac{\mathrm{e}^{\alpha t}m\hbar\omega}{2}\right)^{1/2}\left[\left(\frac{\mathrm{i}\alpha}{2\omega} + 1\right)\hat{a} + \left(\frac{\mathrm{i}\alpha}{2\omega} - 1\right)\hat{a}^\dagger\right] \tag{26}$$

respectively, where $[\hat{x}, \hat{p}] = \mathrm{i}\hbar$ is preserved. Using (25) and (26), we obtain the uncertainty relation as

$$(\Delta x \Delta p)_{n,n} = \left(\frac{\alpha^2}{4\omega^2} + 1\right)^{1/2} \hbar \left(n + \frac{1}{2}\right) \tag{27}$$

and we also find the expectation of the energy operator to be

$$\begin{aligned} E_n &= \mathrm{e}^{-2\alpha t} \frac{\langle \hat{p}^2 \rangle}{2m} + \frac{m}{2}\omega_0^2 \langle \hat{x}^2 \rangle \\ &= \mathrm{e}^{-\alpha t} \frac{\hbar\omega}{2} \left(\frac{\alpha^2}{4\omega^2} + \frac{\omega_0^2}{\omega^2} + 1\right)\left(n + \frac{1}{2}\right) + \mathrm{e}^{-2\alpha t} \frac{p_p^2}{2m} + \frac{m}{2}\omega_0^2 x_{\mathrm{p}}^2. \end{aligned} \tag{28}$$

The energy term including the number n is damped out, but the last term, which constitutes the classical energy by the particular solution, is conserved. Thus, if t is large compared to $1/\alpha$

$$E = \mathrm{e}^{-2\alpha t} \frac{p_p^2}{2m} + \frac{m}{2}\omega_0^2 x_{\mathrm{p}}^2. \tag{29}$$

This result is the classical solution which does not depend on the state. For example, if $f(t) = f_0 \cos \gamma t$, the particular solution is obtained as (A.27). Then

$$\begin{aligned} E &= \frac{m}{2}\dot{x}_{\mathrm{p}}^2 + \frac{m}{2}\omega_0^2 x_{\mathrm{p}}^2 \\ &= \frac{m}{2} A^2 \left[\gamma^2 \sin^2(\gamma t - \delta) + \omega_0^2 \cos^2(\gamma t - \delta)\right]. \end{aligned} \tag{30}$$

Since A depends on α, E also depends on this, but does not damp out.

Second, we obtain the Schrödinger solution for the critical-damped driven harmonic oscillator. By the unitary transformation, the invariant operator (14) can be represented as

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\frac{p_{\mathrm{p}}}{\hbar}x} \hat{I}(p, x, t) \mathrm{e}^{\mathrm{i}\frac{p_{\mathrm{p}}}{\hbar}x} &= \hat{I}(p + p_{\mathrm{p}}, x, t) \\ &= \frac{1}{2m}\left[\frac{m\alpha}{2} \mathrm{e}^{\frac{\alpha}{2}t}(\hat{x} - x_{\mathrm{p}}) + \mathrm{e}^{-\frac{\alpha}{2}t}\hat{p}\right]^2. \end{aligned} \tag{31}$$

Then, (10) becomes

$$\hat{I}(p + p_{\mathrm{p}}, x, t) \mathrm{e}^{-\mathrm{i}\frac{p_{\mathrm{p}}}{\hbar}x} \phi(\lambda, x, t) = \lambda \phi(\lambda, x, t) \mathrm{e}^{-\mathrm{i}\frac{p_{\mathrm{p}}}{\hbar}x}. \tag{32}$$

If we put

$$Q = x - x_{\mathrm{p}} \tag{33}$$

and

$$\Phi(\lambda, Q, t) = \mathrm{e}^{-\mathrm{i}\frac{p_{\mathrm{p}}}{\hbar}x} \phi(\lambda, x, t) \tag{34}$$

then

$$\hat{P} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial Q} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x} = \hat{p} \tag{35}$$

and (32) becomes

$$\hat{I}(P, Q, t)\Phi(\lambda, Q, t) = \lambda \Phi(\lambda, Q, t). \tag{36}$$

If we write

$$q = \sqrt{\frac{m\alpha}{2\hbar}}\, \mathrm{e}^{\frac{\alpha}{2}t} Q \tag{37}$$

then (36) can be represented as the following second-order linear differential equation:

$$\left[\frac{\partial}{\partial q} + \mathrm{i}q\right]^2 \Phi = -\frac{4}{\alpha\hbar}\lambda\Phi. \tag{38}$$

The general solution of (38) for a real number λ is calculated as

$$\Phi(\lambda, q, t) = \mathrm{e}^{-\mathrm{i}\frac{q^2}{2}}\left(A\mathrm{e}^{2\sqrt{\frac{\lambda}{\alpha\hbar}}\mathrm{i}q} + B\mathrm{e}^{-2\sqrt{\frac{\lambda}{\alpha\hbar}}\mathrm{i}q}\right) \tag{39}$$

where both A and B are integral constants.

Substitution of (33), (34) and (37) into (39) gives the eigenfunction of the invariant operator of the critical-damped driven harmonic oscillator (14) as

$$\begin{aligned}\phi(\lambda, x, t) = {}& \mathrm{e}^{\mathrm{i}\frac{p_\mathrm{p}}{\hbar}x}\exp\left[-\mathrm{i}\mathrm{e}^{\alpha t}\frac{m\alpha}{4\hbar}(x - x_\mathrm{p})^2\right] \\ & \times\left(A\exp\left[\mathrm{i}\sqrt{\frac{2m\lambda}{\hbar^2}}\mathrm{e}^{\frac{\alpha}{2}t}(x - x_\mathrm{p})\right] + B\exp\left[-\mathrm{i}\sqrt{\frac{2m\lambda}{\hbar^2}}\mathrm{e}^{\frac{\alpha}{2}t}(x - x_\mathrm{p})\right]\right).\end{aligned} \tag{40}$$

Although the eigenvalue of the under-damped driven harmonic oscillator is discrete, we know that the eigenvalue of the critical-damped driven harmonic oscillator is continuous in (40). Substitution of (40) with (12) into (8) gives

$$\gamma(\lambda, t) = \frac{\alpha t}{4} - \frac{\mathrm{i}}{\hbar}\lambda(t - t_0) - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\left(\mathrm{e}^{-\alpha t}\frac{p_\mathrm{p}^2}{2m} - \frac{m\alpha^2}{8}\mathrm{e}^{\alpha t}x_\mathrm{p}^2\right)\mathrm{d}t. \tag{41}$$

From (12), (40) and (41), the wavefunction of the critical-damped driven harmonic oscillator for the contiguous real constant λ is obtained as

$$\begin{aligned}\Psi(\lambda, x, t) = {}& \exp\left[\frac{\alpha t}{4} - \frac{\mathrm{i}}{\hbar}\lambda(t - t_0) - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\left(\mathrm{e}^{-\alpha t}\frac{p_\mathrm{p}^2}{2m} - \frac{m\alpha^2}{8}\mathrm{e}^{\alpha t}x_\mathrm{p}^2\right)\mathrm{d}t\right] \\ & \times\mathrm{e}^{\mathrm{i}\frac{p_\mathrm{p}}{\hbar}x}\exp\left[-\mathrm{i}\mathrm{e}^{\alpha t}\frac{m\alpha}{4\hbar}(x - x_\mathrm{p})^2\right] \\ & \times\left(A\exp\left[\mathrm{i}\sqrt{\frac{2m\lambda}{\hbar^2}}\mathrm{e}^{\frac{\alpha}{2}t}(x - x_\mathrm{p})\right] + B\exp\left[-\mathrm{i}\sqrt{\frac{2m\lambda}{\hbar^2}}\mathrm{e}^{\frac{\alpha}{2}t}(x - x_\mathrm{p})\right]\right).\end{aligned} \tag{42}$$

The phase of this wavefunction is also represented by the classical action, composed of the classical particular solution, and depends on the past time.

Third, we obtain the Schrödinger solution for the over-damped driven harmonic oscillator. By the unitary transformation, the invariant operator (15), can be represented as

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}\frac{p_\mathrm{p}}{\hbar}\hat{x}}\hat{I}(\hat{p}, \hat{x}, t)\mathrm{e}^{\mathrm{i}\frac{p_\mathrm{p}}{\hbar}\hat{x}} &= \hat{I}(\hat{p} + p_\mathrm{p}, x, t) \\ &= \frac{1}{2}\left(\frac{1}{m}\mathrm{e}^{-\alpha t}\hat{p}^2 + m\omega_0^2\mathrm{e}^{\alpha t}(\hat{x} - x_\mathrm{p})^2 + \frac{\alpha}{2}\left((\hat{x} - x_\mathrm{p})\hat{p} + \hat{p}(\hat{x} - x_\mathrm{p})\right)\right)\end{aligned} \tag{43}$$

which satisfies (32). If we put

$$Q = x - x_\mathrm{p} \tag{44}$$

and

$$\Phi(Q, t) = \mathrm{e}^{-\mathrm{i}\frac{p_\mathrm{p}}{\hbar}x}\phi(x, t) \tag{45}$$

then

$$\hat{P} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial Q} = \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x} = \hat{p} \tag{46}$$

and (36) also holds, i.e.

$$I(P, Q, t)\Phi(Q, t) = \lambda\Phi(Q, t). \tag{47}$$

If we set

$$q = \sqrt{\frac{2m\beta}{\hbar}}\mathrm{e}^{\frac{\alpha}{2}t}Q \tag{48}$$

(47) with (45) can be represented by the following second-order linear differential equation:

$$\frac{\partial^2\Phi}{\partial q^2} + \mathrm{i}\frac{\alpha}{2\beta}q\frac{\partial\Phi}{\partial q} - \frac{\omega_0^2}{4\beta^2}q^2\Phi + \mathrm{i}\frac{\alpha}{4\beta}\Phi = -\frac{1}{\beta\hbar}\lambda\Phi. \tag{49}$$

If we express Φ as

$$\Phi(q) = u(q)\mathrm{e}^{-\mathrm{i}\frac{\alpha}{8\beta}q^2} \tag{50}$$

(49) becomes

$$u'' + \frac{1}{4}q^2u = -\frac{\lambda}{\beta\hbar}u. \tag{51}$$

The general solution of (51) for a real number λ [9, 10] is

$$u_\lambda(q) = A_1\ y_\mathrm{o}\left(q, \frac{\lambda}{\beta\hbar}\right) + A_2\ y_\mathrm{e}\left(q, \frac{\lambda}{\beta\hbar}\right) \tag{52}$$

where A_1 and A_2 are integration constants

$$\begin{aligned} y_\mathrm{o}(q,\eta) = q - \eta\frac{q^3}{3!} &+ \left(\eta^2 - \frac{3}{2}\right)\frac{q^5}{5!} + \left(-\eta^3 + \frac{13}{2}\eta\right)\frac{q^7}{7!} \\ &+ \left(\eta^4 - 17\eta^2 + \frac{63}{4}\right)\frac{q^9}{9!} + \left(-\eta^5 + 35\eta^3 - \frac{531}{4}\eta\right)\frac{q^{11}}{11!} + \cdots \end{aligned} \tag{53}$$

and

$$\begin{aligned} y_\mathrm{e}(q,\eta) = 1 - \eta\frac{q^2}{2!} &+ \left(\eta^2 - \frac{1}{2}\right)\frac{q^4}{4!} + \left(-\eta^3 + \frac{7}{2}\eta\right)\frac{q^6}{6!} \\ &+ \left(\eta^4 - 11\eta^2 + \frac{15}{4}\right)\frac{q^8}{8!} + \left(-\eta^5 + 25\eta^3 - \frac{211}{4}\eta\right)\frac{q^{10}}{10!} + \cdots. \end{aligned} \tag{54}$$

Equations (53) and (54) can be represented by the parabolic cylinder function as [9–11]

$$y_\mathrm{e}(\eta, x) = \frac{2^{\mathrm{i}\eta/2}\Gamma\left(\frac{3}{4} + \mathrm{i}\frac{\eta}{2}\right)}{2^{3/4}\pi^{1/2}}\left[D_{\mathrm{i}\eta-1/2}\left(x\mathrm{e}^{-\mathrm{i}\pi/4}\right) + D_{\mathrm{i}\eta-1/2}\left(-x\mathrm{e}^{-\mathrm{i}\pi/4}\right)\right] \tag{55}$$

and

$$y_\mathrm{o}(\eta, x) = \frac{2^{\mathrm{i}\eta/2}\Gamma\left(\frac{1}{4} + \mathrm{i}\frac{\eta}{2}\right)}{2^{5/4}\pi^{1/2}\mathrm{e}^{\mathrm{i}\pi/4}}\left[D_{-\mathrm{i}\eta-1/2}\left(-x\mathrm{e}^{-\mathrm{i}\pi/4}\right) - D_{-\mathrm{i}\eta-1/2}\left(x\mathrm{e}^{-\mathrm{i}\pi/4}\right)\right] \tag{56}$$

where $D_\mu(z)$ is a solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \left(\mu + \frac{1}{2} - \frac{1}{4}z^2\right)y = 0. \tag{57}$$

From (53) and (54), the recurrence relation of the coefficients $q^n/n!$ for the term a_n is

$$a_{n+2} = -\eta a_n - \tfrac{1}{4}n(n-1)a_{n-2} \tag{58}$$

with

$$a_0 = a_1 = 1 \tag{59}$$

$$a_2 = a_3 = -\eta. \tag{60}$$

Substitution of (33)–(35), (48) and (50) into (52) gives the eigenfunction of the invariant operator of the over-damped driven harmonic oscillator (15) as

$$\begin{aligned} \phi(x,t,\lambda) = \mathrm{e}^{\mathrm{i}\frac{p_\mathrm{p}}{\hbar}x}\exp\left[-\mathrm{i}\frac{m\alpha}{4\hbar}\mathrm{e}^{\alpha t}(x-x_\mathrm{p})^2\right]&\left\{A\ y_\mathrm{o}\left(\sqrt{\frac{2m\beta}{\hbar}}\mathrm{e}^{\frac{\alpha}{2}t}(x-x_\mathrm{p}), \frac{\lambda}{\beta\hbar}\right)\right. \\ &\left. + B\ y_\mathrm{e}\left(\sqrt{\frac{2m\beta}{\hbar}}\mathrm{e}^{\frac{\alpha}{2}t}(x-x_\mathrm{p}), \frac{\lambda}{\beta\hbar}\right)\right\}. \end{aligned} \tag{61}$$

Substitution of (12) with (61) into (8) gives

$$\gamma(\lambda, t) = \frac{\alpha}{4}t - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\left(\mathrm{e}^{-\alpha t}\frac{p_{\mathrm{p}}^2}{2m} - \mathrm{e}^{\alpha t}\frac{m\omega_0^2}{2}x_{\mathrm{p}}^2\right)\mathrm{d}t - \frac{\mathrm{i}}{\hbar}\lambda(t - t_0). \tag{62}$$

From substituting (12) and (61) into (62), the wavefunction of the over-damped driven harmonic oscillator for the contiguous real constant λ is obtained as

$$\begin{aligned}\Psi(x, t, \lambda) = \exp&\left[\frac{\alpha}{4}t - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t}\left(\mathrm{e}^{-\alpha t}\frac{p_{\mathrm{p}}^2}{2m} - \mathrm{e}^{\alpha t}\frac{m\omega_0^2}{2}x_{\mathrm{p}}^2\right)\mathrm{d}t - \frac{\mathrm{i}}{\hbar}\lambda(t - t_0)\right]\mathrm{e}^{\mathrm{i}\frac{p_{\mathrm{p}}}{\hbar}x}\\ &\times \exp\left[-\mathrm{i}\frac{m\alpha}{4\hbar}\mathrm{e}^{\alpha t}(x - x_{\mathrm{p}})^2\right]\left\{A\, y_{\mathrm{o}}\left(\sqrt{\frac{2m\beta}{\hbar}}\mathrm{e}^{\frac{\alpha}{2}t}(x - x_{\mathrm{p}}), \frac{\lambda}{\beta\hbar}\right)\right.\\ &\left.+ B\, y_{\mathrm{e}}\left(\sqrt{\frac{2m\beta}{\hbar}}\mathrm{e}^{\frac{\alpha}{2}t}(x - x_{\mathrm{p}}), \frac{\lambda}{\beta\hbar}\right)\right\}.\end{aligned} \tag{63}$$

The phase of this wavefunction is also represented by the classical action, composed of the classical particular solution, and depends on the past time. If $f(t)$ is applied until the present time t, and t is large compared to $1/\alpha$, the classical solutions are the same for all three cases, but the quantum results are different.

3. Propagator for three cases of the damped driven harmonic oscillator

Since the Schrödinger equation (8) is a linear differential equation, the wavefunction satisfies

$$\Psi(x_2, t_2) = \int_{-\infty}^{\infty} K(x_2, x_1; t_2, t_1)\Psi(x_1, t_1)\,\mathrm{d}x_1 \tag{64}$$

where we call $K(x_2, x_1; t_2, t_1)$ a propagator. If the Hamiltonian is quadratic, the propagator can be obtained as [11]

$$K(x_2, x_1; t_2, t_1) = F(t_2, t_1)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\mathrm{cl}}(x_2, x_1; t_2, t_1)} \tag{65}$$

and

$$F(t_2, t_1) = \sqrt{\frac{\mathrm{i}}{2\pi\hbar}\frac{\partial^2 S_{\mathrm{cl}}}{\partial x_2 \partial x_1}} \tag{66}$$

where $S_{\mathrm{cl}}(x_2, x_1; t_2, t_1)$ is a classical action defined by

$$S_{\mathrm{cl}}(x_2, x_1; t_2, t_1) = \int_{t_1}^{t_2} L(\dot{x}_{\mathrm{c}}, x_{\mathrm{c}}, t)\,\mathrm{d}t. \tag{67}$$

From (2) and (4), the classical action of the damped driven harmonic oscillator can be represented as

$$S_{\mathrm{cl}}(x_2, x_1; t_2, t_1) = \frac{m}{2}\mathrm{e}^{\alpha t}\dot{x}x\bigg|_{t_1}^{t_2} + \frac{m}{2}\int_{t_1}^{t_2}\mathrm{e}^{\alpha t}xf(t)\,\mathrm{d}t. \tag{68}$$

Using (A.1)–(A.8) from the appendix with (68), we can calculate the classical action of the under-damped driven harmonic oscillator as

$$\begin{aligned}S_{\mathrm{cl}} = \frac{m}{2}&\left(\frac{\omega}{\sin\omega T}\left[-2\mathrm{e}^{\frac{\alpha}{2}(t_1+t_2)}x_1x_2\right.\right.\\ &\left.\left.+\left(\mathrm{e}^{\alpha t_2}x_2^2 + \mathrm{e}^{\alpha t_1}x_1^2\right)\cos\omega(t_2 - t_1)\right] + \frac{\alpha}{2}\left(\mathrm{e}^{\alpha t_1}x_1^2 - \mathrm{e}^{\alpha t_2}x_2^2\right)\right)\end{aligned}$$

$$+ 2x_1 \int_{t_1}^{t_2} \frac{\mathrm{e}^{\frac{\alpha}{2}(t'+t_1)} \sin\omega(t_2 - t')}{\sin\omega(t_2 - t_1)} f(t')\,\mathrm{d}t'$$
$$+ 2x_2 \int_{t_1}^{t_2} \frac{\mathrm{e}^{\frac{\alpha}{2}(t'+t_2)} \sin\omega(t' - t_1)}{\sin\omega(t_2 - t_1)} f(t')\,\mathrm{d}t'$$
$$+ 2 \int_{t_1}^{t_2} \mathrm{d}t \int_{t_1}^{t} \mathrm{d}t' \frac{\mathrm{e}^{\frac{\alpha}{2}(t'+t)} \sin\omega(t' - t_1) \sin\omega(t - t_2)}{\sin\omega(t_2 - t_1)} f(t') f(t). \quad (69)$$

Substituting (69) into (66), the amplitude of its propagator becomes

$$F(t_2, t_1) = \left[\frac{m\omega \mathrm{e}^{\frac{\alpha}{2}(t_1+t_2)}}{2\pi \mathrm{i}\hbar \sin\omega(t_2 - t_1)}\right]^{1/2}. \quad (70)$$

Thus, we can obtain the propagator of the under-damped driven harmonic oscillator by combining (69) and (70) with (65).

Using (A.9)–(A.16) with (68), we can calculate the classical action of the critical-damped driven harmonic oscillator as

$$S_{\mathrm{cl}} = \frac{m}{2}\left[\frac{\alpha}{2}\left(\mathrm{e}^{\alpha t_1} x_1^2 - \mathrm{e}^{\alpha t_2} x_2^2\right) + \frac{\mathrm{e}^{\alpha t_2} x_2^2 - \mathrm{e}^{\alpha t_1} x_1^2}{(t_2 - t_1)}\right.$$
$$+ 2x_1 \int_{t_1}^{t_2} \mathrm{e}^{\frac{\alpha}{2}(t'+t_1)} \frac{(t_2 - t')}{(t_2 - t_1)} f(t')\,\mathrm{d}t'$$
$$+ 2x_2 \int_{t_1}^{t_2} \mathrm{e}^{\frac{\alpha}{2}(t'+t_2)} \frac{(t' - t_1)}{(t_2 - t_1)} f(t')\,\mathrm{d}t'$$
$$\left. - 2 \int_{t_1}^{t_2} \mathrm{d}t \int_{t_1}^{t} \mathrm{d}t' \mathrm{e}^{\frac{\alpha}{2}(t'+t)} \frac{(t - t_1)(t_2 - t')}{(t_2 - t_1)} f(t') f(t)\right]. \quad (71)$$

Substitution of (71) into (66), the amplitude of its propagator is obtained as

$$F(t_2, t_1) = \left[\frac{m \mathrm{e}^{\frac{\alpha}{2}(t_2+t_1)}}{2\pi \mathrm{i}\hbar (t_2 - t_1)}\right]^{1/2}. \quad (72)$$

Thus, the propagator of the critical-damped driven harmonic oscillator can be found by combining (71) and (72) with (65).

Using (A.17)–(A.24) with (68), we can calculate the classical action of the over-damped driven harmonic oscillator as

$$S_{\mathrm{cl}} = \frac{m}{2}\left(\frac{\beta}{\sinh\beta(t_2 - t_1)}\left[-2\mathrm{e}^{\frac{\alpha}{2}(t_1+t_2)} x_1 x_2\right.\right.$$
$$\left.\left. + \left(\mathrm{e}^{\alpha t_2} x_2^2 + \mathrm{e}^{\alpha t_1} x_1^2\right)\cosh\omega(t_2 - t_1)\right] + \frac{\alpha}{2}\left(\mathrm{e}^{\alpha t_1} x_1^2 - \mathrm{e}^{\alpha t_2} x_2^2\right)\right)$$
$$+ 2x_1 \int_{t_1}^{t_2} \frac{\mathrm{e}^{\frac{\alpha}{2}(t'+t_1)} \sinh\beta(t_2 - t')}{\sinh\beta(t_2 - t_1)} f(t')\,\mathrm{d}t'$$
$$+ 2x_2 \int_{t_1}^{t_2} \frac{\mathrm{e}^{\frac{\alpha}{2}(t'+t_2)} \sinh\beta(t' - t_1)}{\sinh\beta(t_2 - t_1)} f(t')\,\mathrm{d}t'$$
$$+ 2 \int_{t_1}^{t_2} \mathrm{d}t \int_{t_1}^{t} \mathrm{d}t' \frac{\mathrm{e}^{\frac{\alpha}{2}(t'+t)} \sinh\beta(t' - t_1) \sinh\beta(t - t_2)}{\beta \sinh\beta(t_2 - t_1)} f(t') f(t). \quad (73)$$

Substituting (73) into (66), the amplitude of its propagator is obtained as

$$F(t_2, t_1) = \left[\frac{m\beta \mathrm{e}^{\frac{\alpha}{2}(t_2+t_1)}}{2\pi \mathrm{i}\hbar \sinh\beta(t_2 - t_1)}\right]^{1/2}. \quad (74)$$

Thus, the propagator of the over-damped driven harmonic oscillator can be found by combining (73) and (74) with (65).

Next, we treat the propagator using the wavefunction. If the energy eigenvalue is discrete, the propagator can be represented by

$$K(x_2, x_1; t_2, t_1) = \sum_n \Psi_n^*(x_2, t_2)\Psi_n(x_1, t_1). \tag{75}$$

If we use Mehler's formula

$$\sqrt{1-z^2}\mathrm{e}^{\frac{2XY-X^2-Y^2}{1-z^2}} = \mathrm{e}^{-X^2-Y^2}\sum_0^\infty \frac{z^n}{2^n n!}H_n(X)H_n(Y) \tag{76}$$

and the wavefunction of the under-damped driven harmonic oscillator, equation (24), (75) gives its propagator as (65) combined with (69) and (70).

If the energy eigenvalue is continuous, the propagator can be obtained as

$$K(x_2, x_1; t_2, t_1) = \int \Psi^*(x_2, t_2, \lambda)\Psi(x_1, t_1, \lambda)\,\mathrm{d}\lambda. \tag{77}$$

If we use the formula

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\lambda(X-Y)}\mathrm{e}^{\mathrm{i}\lambda^2}\,\mathrm{d}\lambda = \sqrt{\frac{\pi}{\mathrm{i}}}\mathrm{e}^{-(X-Y)^2} \tag{78}$$

and the wavefunction of the critical-damped driven harmonic oscillator, equation (42), (77) gives the propagator as (65) combined with (71) and (72). If we use Erdelyi's formula [9–11]

$$\begin{aligned}
&\frac{\mathrm{i}}{2}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\left[D_\mu(x)D_{-\mu-1}(\mathrm{i}y) + D_\mu(-x)D_{-\mu-1}(-\mathrm{i}y)\right]\frac{t^{-\mu-1}}{\sin -\mu\pi}\,\mathrm{d}\mu \\
&\quad = -\sqrt{2\pi}(1+t^2)^{-1/2}\exp\left(\frac{1}{4}\,\frac{1-t^2}{1+t^2}(x^2+y^2) + \mathrm{i}\frac{txy}{1+t^2}\right) \\
&\quad (-1 < c < 0\,,\ |\arg\ t| < \pi/2)
\end{aligned} \tag{79}$$

and the wavefunction of the over-damped driven harmonic oscillator, equation (63), with (55) and (56), (77) gives the propagator as (65) combined with (73) and (74).

4. Summary and conclusions

In this section we summarize and discuss the results obtained in the previous sections. In section 2, we reviewed the classical-damped driven harmonic oscillator. We selected the Kanai Hamiltonian among the innumerable ones which give the same classical-equation motion. It is well known that they are connected by a canonical transformation with each other [7]. We know that the three cases of the damped driven harmonic oscillator, i.e., under-, critical- and over-damping, can be determined by the damping constant and frequency. The under-damped driven harmonic oscillator has oscillatory motion, but the critical- and over-damped driven harmonic oscillators do not. In order to calculate the particular solutions, we obtained the Green functions with the one- and two-point boundary conditions for the three cases in the appendix. With these, we obtained the general solutions of the three systems. It is well known that the solutions of the systems become the particular solutions when t is sufficiently large compared to $1/\alpha$.

We treated the wavefunctions of the three cases of the damped driven harmonic oscillator. The quantum Hamiltonian can be found as a corresponding classical one in which the canonical variables are replaced by the corresponding quantum operator. Although there is one Schrödinger equation for the damped driven harmonic oscillator, three classical solutions emerge depending on the damping constant and the frequency. To obtain the wavefunction, we determined the quadratic invariant operators and calculated their eigenfunction. Here, we showed that for the under-damping case, the invariant operator can be expressed in terms of creation and the annihilation operators, and the differential equation for solving the eigenvalue equation is related to the time-independent Schrödinger equation of the simple harmonic oscillator. Thus, its eigenfunction was obtained as the wavefunction of the simple harmonic oscillator.

For the critical-damping case, the differential equation for solving the eigenvalue equation is related to the Schrödinger equation of the free particle. Thus, its eigenfunction was obtained as the wavefunction of the free particle. For the over-damping case, the differential equation for solving the eigenvalue equation is related to the Schrödinger equation of the harmonic parabola potential system [10]. Thus, its eigenfunction is obtained as the wavefunction of the harmonic parabola potential system. We found that the eigenvalues of the oscillatory system, i.e., the under-damping system, are discrete, but those of the non-oscillatory systems, i.e., the critical- and the over-damping systems, are continuous. The phase factors for the three cases can also be obtained by the Schrödinger equation and their wavefunctions, related to their classical actions.

In section 3, we evaluated the propagator for the three cases of the damped driven harmonic oscillator. It is well known that the propagator of the quadratic Hamiltonian has an exponent composed of the classical action. Thus, with the calculation of the classical actions of the three cases, we obtained both the exponent and amplitude of their propagators. We checked the calculation of the propagator of the under-damped driven harmonic oscillator with its wavefunction using Mehler's formula. The calculations of the propagators of the critical- and over-damped driven harmonic oscillators were also checked with their wavefunctions using Gauss' and Erdelyi's formulae, respectively.

The non-oscillatory systems, i.e., the critical- and over-damped driven harmonic oscillators, look like the driven free-particle and harmonic parabola potential systems, respectively. Their classical motion is monotonic. Thus, their quantum behaviour is treated with appropriate boundary conditions. Since their classical results are canonically related to the driven free-particle and harmonic parabola potential systems, respectively, their quantum behaviour with appropriate boundary conditions also has an associated unitary relation. Using our results, in the near future, we plan to carry out quantum mechanical studies of these non-oscillatory systems with appropriate boundary conditions.

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Appendix. Classical survey of a damped driven harmonic oscillator

There are three general cases of the solution of (4): the under-, critical- and over-damped driven harmonic oscillators. Let us first consider the under-damped case, equation (5), which is

$$-\omega^2 \equiv \frac{\alpha^2}{4} - {\omega_0}^2 < 0. \tag{A.1}$$

The general solution of (4) is

$$x(t) = \mathrm{e}^{-\frac{\alpha}{2}t}(A_\mathrm{u}\mathrm{e}^{\mathrm{i}\omega t} + B_\mathrm{u}\mathrm{e}^{-\mathrm{i}\omega t}) + x_\mathrm{p}(t) \tag{A.2}$$

where the particular solution $x_{\rm p}(t)$ is (if the boundary conditions are of the two-point type)

$$x_{\rm p}(t) = \int_{t_1}^{t_2} G(t,t')f(t')\,{\rm d}t'. \tag{A.3}$$

Here the Green function $G(t,t')$ is

$$G(t,t') = \begin{cases} G_1(t,t') & \text{for} \quad t > t' \\ G_2(t,t') & \text{for} \quad t < t' \end{cases} \tag{A.4}$$

with

$$G_1(t,t') = \frac{{\rm e}^{\frac{\alpha}{2}(t'-t)}\sin\omega(t_1-t')\sin\omega(t-t_2)}{\omega\sin\omega(t_1-t_2)} \tag{A.5}$$

$$G_2(t,t') = \frac{{\rm e}^{\frac{\alpha}{2}(t'-t)}\sin\omega(t_2-t')\sin\omega(t-t_1)}{\omega\sin\omega(t_2-t_1)}. \tag{A.6}$$

If the boundary condition is of the one-point type, then

$$x_{\rm p}(t) = \int_{t_0}^{t} G(t,t')f(t')\,{\rm d}t' \tag{A.7}$$

where

$$G(t,t') = \frac{1}{\omega}\,{\rm e}^{-\frac{\alpha}{2}(t-t')}\sin\omega(t-t'). \tag{A.8}$$

The second case, i.e., the critical-damped harmonic oscillator, equation (6), is

$$\frac{\alpha^2}{4} - {\omega_0}^2 = 0. \tag{A.9}$$

The general solution of (4) is

$$x(t) = {\rm e}^{-\frac{\alpha}{2}t}(A_{\rm c} + B_{\rm c}t) + x_{\rm p}(t) \tag{A.10}$$

where the particular solution $x_{\rm p}(t)$ is (if the boundary conditions are of the two-point type)

$$x_{\rm p}(t) = \int_{t_1}^{t_2} G(t,t')f(t')\,{\rm d}t'. \tag{A.11}$$

Here the Green function $G(t,t')$ is

$$G(t,t') = \begin{cases} G_1(t,t'), & \text{for} \quad t > t' \\ G_2(t,t') & \text{for} \quad t < t' \end{cases} \tag{A.12}$$

with

$$G_1(t,t') = \frac{{\rm e}^{\frac{\alpha}{2}(t_1-t')}(t_1-t')(t-t_2)}{(t_1-t_2)} \tag{A.13}$$

$$G_2(t,t') = \frac{{\rm e}^{\frac{\alpha}{2}(t'-t)}(t_2-t')(t-t_1)}{(t_2-t_1)}. \tag{A.14}$$

If the boundary condition is of the one-point type, then the particular solution is

$$x_{\rm p}(t) = \int_{t_0}^{t} G(t,t')f(t')\,{\rm d}t' \tag{A.15}$$

where

$$G(t,t') = (t-t'){\rm e}^{-\frac{\alpha}{2}(t-t')}. \tag{A.16}$$

For the third case, i.e., the over-damped driven harmonic oscillator, equation (7), is

$$\beta^2 = \frac{\alpha^2}{4} - {\omega_0}^2 > 0. \tag{A.17}$$

The general solution of (4) is

$$x(t) = \mathrm{e}^{-\frac{\alpha}{2}t}(A_\mathrm{o}\mathrm{e}^{\beta t} + B_\mathrm{o}\mathrm{e}^{-\beta t}) + \int_{t_1}^{t_2} G(t, t')f(t')\,\mathrm{d}t' \tag{A.18}$$

where the particular solution $x_\mathrm{p}(t)$ is, if the boundary conditions are of the two-point type

$$x_\mathrm{p}(t) = \int_{t_1}^{t_2} G(t, t')f(t')\,\mathrm{d}t'. \tag{A.19}$$

Here the Green function $G(t, t')$ is

$$G(t, t') = \begin{cases} G_1(t, t') & \text{for} \quad t > t' \\ G_2(t, t') & \text{for} \quad t < t' \end{cases} \tag{A.20}$$

with

$$G_1(t, t') = \frac{\mathrm{e}^{\frac{\alpha}{2}(t'-t)} \sinh\beta(t_2 - t)\sinh\beta(t_1 - t')}{\beta\sinh\beta T} \tag{A.21}$$

$$G_2(t, t') = \frac{\mathrm{e}^{\frac{\alpha}{2}(t'-t)} \sinh\beta(t_2 - t')\sinh\beta(t_1 - t)}{\beta\sinh\beta T}. \tag{A.22}$$

If the boundary condition is of the one-point type, then

$$x_\mathrm{p}(t) = \int_{t_0}^{t} G(t, t')f(t')\,\mathrm{d}t' \tag{A.23}$$

where

$$G(t, t') = \frac{1}{\beta}\,\mathrm{e}^{-\frac{\alpha}{2}(t-t')}\sinh\beta(t - t'). \tag{A.24}$$

We know that if $f(t)$ is applied until the present time t, the particular solution $x_\mathrm{p}(t)$ does not depend on the damping factor in the three cases of the damped driven harmonic motion. Thus, if t is large compared to $1/\alpha$

$$x(t) = x_\mathrm{p}(t) \tag{A.25}$$

which does not depend on the initial condition. Although the form of the Green function is different for each of the three cases, their particular solutions have the same form. For example, if the driving force is

$$f(t) = f_0\cos(\gamma t) \tag{A.26}$$

the particular solution can be calculated as

$$x_\mathrm{p}(t) = A\cos(\gamma t - \delta) \tag{A.27}$$

where, in the three cases of the damped driven harmonic motion, the amplitude A is

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \gamma^2)^2 + \gamma^2\alpha^2}} \tag{A.28}$$

and the phase difference between the driving force and the resultant motion is

$$\delta = \tan^{-1}\left(\frac{\alpha\gamma}{\omega_0^2 - \gamma^2}\right). \tag{A.29}$$

If the driving force is applied in just the finite time interval $[t_0, t_1]$, the particular solution is

$$x_\mathrm{p}(t) = \mathrm{e}^{-\frac{\alpha}{2}t} \int_{t_0}^{t_1} \frac{1}{\omega}\, \mathrm{e}^{\frac{\alpha}{2}t'} \sin\omega(t-t') f(t')\, \mathrm{d}t' \tag{A.30}$$

$$x_\mathrm{p}(t) = \mathrm{e}^{-\frac{\alpha}{2}t} \int_{t_0}^{t_1} (t-t')\, \mathrm{e}^{\frac{\alpha}{2}t'} f(t')\, \mathrm{d}t' \tag{A.31}$$

and

$$x_\mathrm{p}(t) = \mathrm{e}^{-\frac{\alpha}{2}t} \int_{t_0}^{t_1} \frac{1}{\beta}\, \mathrm{e}^{\frac{\alpha}{2}t'} \sinh\beta(t-t') f(t')\, \mathrm{d}t' \tag{A.32}$$

for the three cases of the damped driven harmonic motion, respectively. Here, we know that if t is large compared to $1/\alpha$, $x_\mathrm{p}(t)$ is also damped out and becomes zero, as does the complimentary solution, i.e., the first term on the right-hand side of (A.18).

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